Section 4.1: Antiderivatives and Indefinite Integration

Antiderivatives

An *antiderivative* of $f(x)$ is a function whose derivative is $f(x)$.

- Differentiation and antidifferentiation are "inverse" operations of each other. That is, they undo the effect of each other.
- The important exception to this rule is that if you differentiate a function and then take its antiderivative, you do not get the original function, but the original function plus an arbitrary constant.
- If $F(x)$ is an antiderivative of $f(x)$, so is $G(x) = F(x) + C$, where C is a constant.

The antiderivative is indicated by the integral symbol ∫. The antiderivative of the function $f(x) = x^2$ is denoted as follows:

$$\int x^2\,dx = \frac{x^3}{3} + C$$

It is often necessary to rewrite an integrand in terms of known derivatives of known functions before taking the antiderivative (see examples 5 and 6 on page 252).

Differential equations

An equation that involves x, $y(x)$ and derivatives of y (dy/dx, d^2y/dy^2, etc.) is called a differential equation in x and y.

For example, $y' = 4x$ is a differential equation whose solution is $y = 2x^2 + C$.

To solve a differential equation, first rewrite it in differential form (i.e. $dy = f'(x)dx$) and take the antiderivative of both sides.

Section 4.2: Area

Sigma (∑, the Greek equivalent of the letter 'S') is used to express sums in a compact form. For example, the sum of the series $a_i = i^2$ from $i = 1$ to 5 would be represented by:

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{i=1}^{5} i^2$$

This would be read as "the sum of "a" sub "i" for "i" from "1" to "n"

Here are four useful summation formulas involving ∑:

$$\sum_{i=1}^{n} c = c + c + c + \cdots + c = cn$$
$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n+1)(n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Area of a plane region

The area between a curve and the *x*-axis can be approximated by dividing the interval over which the area is to be calculated into subintervals Δx; constructing a rectangle upon each interval whose width is the interval and height $f(c_i)$ is a value of the function evaluated at a point c_i in that subinterval; and adding up the areas of the rectangles. [See Fig. 4.9 (pg. 262)]

$$\boxed{\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x}$$

If the interval over which the area is to be calculated is [*a*, *b*] and the number of subintervals is *n*, then $\Delta x = \dfrac{b-a}{n}$.

The points c_i may be chosen so that the rectangles intersect the curve on their left sides, their right sides, or anywhere in between.

- If c_i are chosen so that the tops of rectangles are always ≤ $f(x)$ (i.e. are inscribed), the area of the rectangles is called a *lower sum* and is ≤ the area under the curve. [See Fig. 4.12 (pg. 263)]

- If c_i are chosen so that the tops of rectangles are always ≥ $f(x)$ (i.e. are circumscribed), the area of the rectangles is called an *upper sum* and is ≥ the area under the curve.

- If the right endpoint of each subinterval is the chosen value for c_i, then c_i becomes simply $a + i\Delta x$ where *a* is the point at which the summing starts.

The limits of the upper and lower sums *coincide* as the number of rectangles (denoted *n*) approaches infinity. Therefore, according to the Squeeze Theorem (Theorem 1.8), they each limit is equal to the true area under the curve.

Hence,

$$\boxed{\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x}$$

Procedure for calculating areas using limits:

1. Write an expression for Δx using $\Delta x = \frac{b-a}{n}$
2. Write an expression for c_i using $c_i = a + \Delta x$
3. Calculate $f(c_i)$
4. Set up the area expression and simplify algebraically
5. Use the summation formulas to rewrite in terms of n
6. Evaluate the limit

For an example of how this equation can be used to calculate the area under a curve, see Examples 5 and 6 on page 266.
